Sovereign Partial Default in Continuous Time

Sangdong Kim   Gabriel Mihalache

The Ohio State University

2023 A. Stockman University of Rochester Alumni Conference

What We Do

- Continuous time analysis and solution methods for the *partial default* quantitative theory of Arellano, Mateos-Planas, and Ríos-Rull (2023, JPE)

- Partial default theory consistent with…
  - endogenous length of default crisis,
  - exiting the crisis with a high debt level compared to outset, arrears,
  - implicit seniority among creditors, Schleg, Trebesch, and Wright (2019)

- Computation with
  - Upwind finite difference scheme, Achdou et al. (2022, ReStud)
  - Deep neural network, Maliar, Maliar, and Winant (2021, JME)

Partial Default

- Sovereign chooses with discretion what share of the *due debt service payment* to make. Default on flow, not on stock.
- The share of payment not made accumulates as *arrears*, extra debt.
- Convex penalty function of default with discontinuity at 0, *inaction region*,
- No "market exclusion," sovereign can issue new bond units at all times, at prices at which lenders break even in expectation.

In discrete time:

$$
\begin{aligned}
c_t &= \phi(d_t) z_t - (1 - d_t) b_t + q_t \ell_t \\
b_{t+1} &= (1 - \delta) b_t + \kappa d_t b_t + \ell_t
\end{aligned}
$$

The Sovereign

$$
\begin{aligned}
V(B_0, z_0) = \max_{\{c_t, d_t\}_{t\in[0,\infty]}} \; & \mathbb{E}_0 \left\{ \int_0^\infty e^{-\rho t} u(c_t) dt \right\} \\
\text{s.t.} \quad & c_t = \phi\left(d_t, z_t\right) e^{z_t} - \left(1 - d_t\right)\left(\delta + \lambda\right) B_t + q_t \ell_t \\
& \frac{dB_t}{dt} = -\delta B_t + \kappa\left(\delta + \lambda\right) d_t \, B_t + \ell_t
\end{aligned}
$$

The Sovereign

$$V(B_0, z_0) = \max_{\{c_t, d_t\}_{t \in [0,\infty]}} \mathbb{E}_0 \left\{ \int_0^\infty e^{-\rho t} u(c_t) dt \right\}$$

$$\text{s.t.} \quad c_t = \phi\left(d_t, z_t\right) e^{z_t} - \left(1 - d_t\right)\left(\delta + \lambda\right) B_t + q_t \ell_t$$

$$\frac{dB_t}{dt} = -\delta B_t + \kappa\left(\delta + \lambda\right) d_t \, B_t + \ell_t$$

Consumption c_t:

- GDP: $\phi\left(d_t, z_t\right) e^{z_t}$
- Debt service payment (minus): $\left(1 - d_t\right)\left(\delta + \lambda\right) B_t$
- New issuance proceeds: $q_t \ell_t$

The Sovereign

$$V(B_0, z_0) = \max_{\{c_t, d_t\}_{t\in[0,\infty]}} \mathbb{E}_0 \left\{ \int_0^\infty e^{-\rho t} u(c_t) dt \right\}$$

$$\text{s.t.} \quad c_t = \phi(d_t, z_t) e^{z_t} - (1 - d_t)(\delta + \lambda) B_t + q_t \ell_t$$

$$\frac{dB_t}{dt} = -\delta B_t + \kappa(\delta + \lambda) d_t B_t + \ell_t$$

Drift of debt dB_t/dt:

- Maturing debt (minus): δB_t
- Arrears: $\kappa(\delta + \lambda) d_t B_t$
- New issuance: ℓ_t

The Sovereign: HJB and FOCs

$$\rho V(B,z) = \max_{c,d\in[0,1]} \left\{ u(c) + S(B,z,c,d,q) V_B(B,z) - \mu z V_z(B,z) + \frac{\sigma^2}{2} V_{zz}(B,z) \right\} \quad (1)$$

$$S(B,z,c,d,q) \equiv \frac{c - \phi\left(d,z\right) e^z}{q(B,z)} + \left[\left(\frac{1}{q(B,z)} + \left(\kappa - \frac{1}{q(B,z)} \right) d \right) (\delta + \lambda) - \delta \right] B$$

The Sovereign: HJB and FOCs

$$\rho V(B,z) = \max_{c,d\in[0,1]} \left\{ u(c) + S(B,z,c,d,q) V_B(B,z) - \mu z V_z(B,z) + \frac{\sigma^2}{2} V_{zz}(B,z) \right\} \quad (1)$$

$$S(B,z,c,d,q) \equiv \frac{c - \phi\,(d,z)\,e^z}{q(B,z)} + \left[\left(\frac{1}{q(B,z)} + \left(\kappa - \frac{1}{q(B,z)} \right) d \right) (\delta + \lambda) - \delta \right] B$$

FOCs:

$$c^*(B,z) = u_c^{-1} \left(-\frac{V_B(B,z)}{q(B,z)} \right) \quad (2)$$

$$d^*_{\text{int}}(B,z) = \min \left\{ 1,\ \phi_d^{-1} \left((1 - \kappa\, q(B,z))\,(\delta + \lambda) \frac{B}{e^z},\ z \right) \right\} \quad (3)$$

International Lenders

$$q_t = \mathbb{E}_t \int_t^{\infty} e^{-(r+\delta)(s-t)+\int_t^s \kappa(\delta+\lambda)d_\tau d\tau}(\delta+\lambda)(1-d_s)d_s$$

$$\xi(d^*(B,z))q(B,z) = (1-d^*(B,z))(\lambda+\delta) + \tilde{S}(B,z)q_B(B,z) - \mu z q_z(B,z) + \frac{\sigma^2}{2}q_{zz}(B,z) \tag{4}$$

With,

- Effective discount rate, inclusive of arrears: $\xi(d) \equiv r + \delta - \kappa(\delta+\lambda)d$
- Equilibrium drift of debt: $\tilde{S}(B,z)$

Equilibrium

A *Markov Perfect Equilibrium* consist of

- the sovereign's value function $V(B, z)$,
- policy functions for consumption and default, $c^*(B, z)$ and $d^*(B, z)$, and
- the bond price function $q(B, z)$,

such that

- given q and V, the policies satisfy FOCs (2) and (3),
- given q and policies, the sovereign's value satisfies the HJB equation (1), and
- given policy functions, the bond price satisfies equation (4).

Ergodic Distribution (KFE)

Kolmogorov Forward Equation

$$\frac{\partial}{\partial t} f(B,z,t) = -\frac{\partial}{\partial B}\left[\tilde{S}(B,z)f(B,z,t)\right] + \frac{\partial}{\partial z}\left[\mu z f(B,z,t)\right] + \frac{\sigma^2}{2}\frac{\partial^2}{\partial z^2} f(B,z,t)$$

Ergodic distribution f^* satisfies

$$0 = -\frac{\partial}{\partial B}\left[\tilde{S}(B,z)f^*(B,z)\right] + \frac{\partial}{\partial z}\left[\mu z f^*(B,z)\right] + \frac{\sigma^2}{2}\frac{\partial^2}{\partial z^2} f^*(B,z)$$

Calibration

Parameter	Value	Comment
Preferences		
ν	2.0	AMPRR
ρ	0.047	Implied discount rate
Debt		
δ	0.13	Macaulay duration
κ	0.7	Partial default haircut
r	0.039	Ct. compounding rate
λ	r	Normalization
Endowment Process		
μ	0.221	Time aggregation of
σ	0.062	AMPRR AR(1)
Default Penalty		
γ_0	0.0476	AMPRR
γ_1	2.0	AMPRR
γ_2	0.12	AMPRR
$\tilde{z}$	-0.062	AMPRR

$$dz_t = -\mu z_t dt + \sigma dW_t$$

(Ornstein-Uhlenbeck with $[\underline{z}, \bar{z}]$ barriers)

$$u(c_t) = \begin{cases} \frac{c_t^{1-\nu}}{1-\nu} & \text{if } \nu \neq 1 \\ \log c_t & \text{if } \nu = 1 \end{cases}$$

$$\phi(d_t, z_t) = \left(1 - \gamma_0 d_t^{\gamma_1}\right) \times \left[1 - (z_t - \tilde{z})\,\gamma_2 \mathbb{1}_{\{d_t > 0 \text{ and } z_t > \tilde{z}\}}\right]$$

Computation: Two Methods

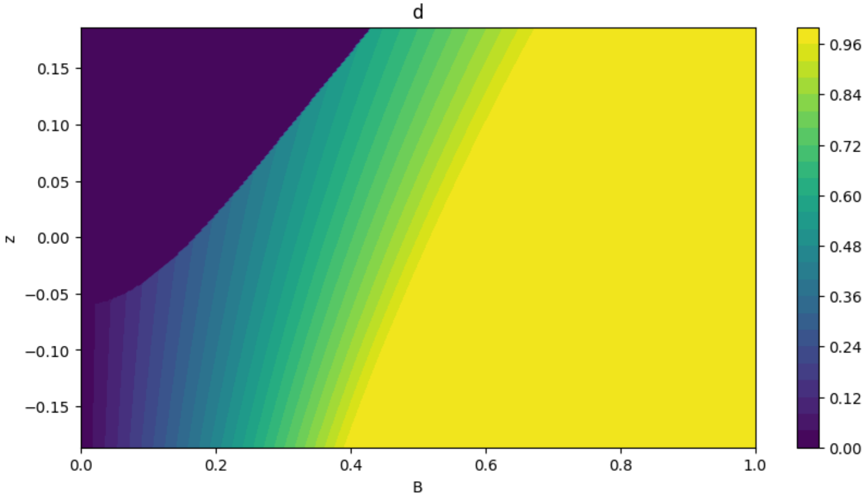
Today, *deep neural network*:

- Neural network inputs $< B, z >$, outputs $< V, q, c >$
- Minimize minibatch residuals of HJB for V, HJB for q, and FOC for c
- Stochastic gradient descent
- Method amenable to extensions with many state variables

Work in progress, *upwind finite difference scheme*:

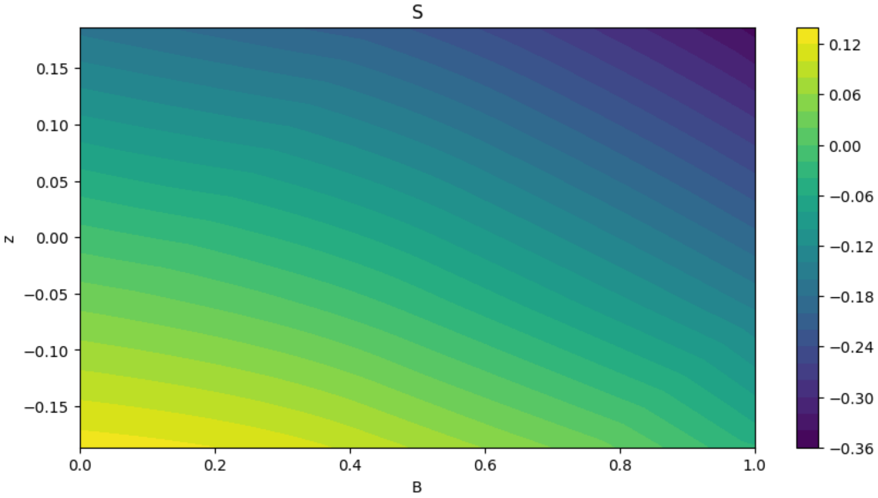
- Fast and precise
- Use in literature on traditional default models in continuous time, Hurtado, Nuño, and Thomas (2023, JEEA) and Borstein (2020, JEDC)

Preliminary Results: Default Policy



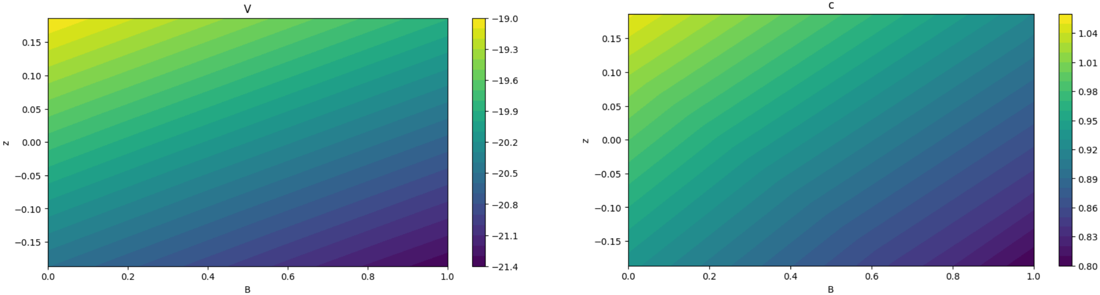
- Inaction zone ($d = 0$) dark blue, no debt service ($d = 1$) in yellow. Intermediate intensities in blue-green range.

Preliminary Results: Drift of B



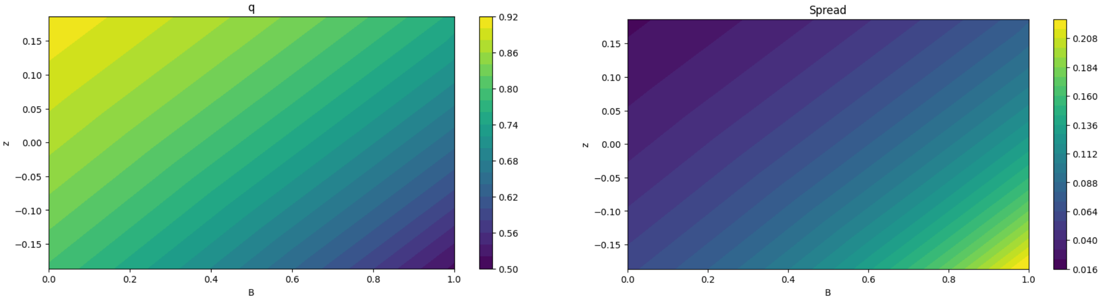
- Borrow aggressively when debt is low and endowment is low, reduce debt at high debt and/or high endowment. Keep B in place at $S = 0$, mid-green.

Preliminary Results: Value Function and Consumption



- Value and consumption increasing in endowment (z) and decreasing in debt (B)

Preliminary Results: Bond Price and Spreads



- Spreads decreasing in endowment (z) and increasing in debt (B)