Sovereign Partial Default in Continuous Time*

Sangdong Kim
The Ohio State University

Gabriel Mihalache
The Ohio State University

February 2024

Abstract

We formulate and solve a tractable, continuous time version of the sovereign *partial default* model of Arellano, Mateos-Planas, and Ríos-Rull (2023). We compute our model using the implicit upwind finite difference scheme. We show that our formulation allows for a tight characterization of debt and default dynamics, as well as the length and severity of crisis events.

Keywords: sovereign debt, partial default, continuous time
JEL classification: E44, F34, H63

*We are thankful to Narayana Kocherlakota, Galo Nuno, and seminar participants at Ohio State and the University of Rochester for many insightful questions and comments. *Contact information:* sangdongkim.osu@gmail.com; mihalache@gmail.com (corresponding author)

1 Introduction

This paper proposes a tractable, continuous time model of sovereign partial default, building on the theoretic-quantitative work of Arellano, Mateos-Planas, and Ríos-Rull (2023). We solve our model using the implicit upwind finite difference scheme as in Hurtado, Nuño, and Thomas (2023).

The standard theory of sovereign default, under incomplete markets and lack of commitment, as introduced by Eaton and Gersovitz (1981) and quantified by Aguiar and Gopinath (2006) and Arellano (2008), assumes that default is a discrete choice which is taken with respect to the entire stock of outstanding debt.[1] By choosing to default, the sovereign transitions into a separate regime, characterized by international financial market exclusion and specific resource or utility costs. Eventually, the sovereign returns to market access with either no outstanding obligations or a debt level lower than that at the time of default, as determined by renegotiation with its lenders.[2] In contrast, the theory of *partial* default developed by Arellano, Mateos-Planas, and Ríos-Rull (2023) is consistent with the observations that sovereigns often discriminate between their creditors, selectively suspending debt service payments on only some instruments, and that debt levels rise during default crises, including due to the issuance of new obligations, with countries sometimes owing more at the end of such episodes than at their outbreak. In this theory, as in the data, missed payments accumulate as arrears and further deteriorate the fiscal sustainability outlook. Arellano, Bai, and Mihalache (2023) employ this theory to study the budgetary impact of the COVID-19 epidemic in emerging markets, its interaction with costly mitigation measures, and to simulate debt relief programs and counterfactuals.

Our paper relates to the recent body of work on quantitative sovereign default models in continuous time. Bornstein (2020) compares discrete and continuous time versions of the textbook default model and finds that computation is faster in continuous time.[3] He identifies a *painful deleveraging* effect specific to such settings, where the stock of debt is slow-moving, and provides results on the relationship between equilibrium default and debt maturity, including the possibility of positive default risk with short-term debt if the shock is a jump process. Tourre (2017) focuses on the macro-financial implications of the standard, complete default model, by studying a setting with recursive utility and a rich pricing kernel for international lenders, which allows him to connect the model with the cross-country comovement of spreads in the data. Finally, Hurtado, Nuño, and Thomas (2023) is the paper closest to our work. They study a standard sovereign default model, with long-term debt denominated in local currency and thus exposed to the exchange rate depreciation induced by the government's choice of inflation. This use of

1. Textbook treatments of this sovereign default framework are provided by Uribe and Schmitt-Grohé (2017) and Aguiar and Amador (2021). The handbook chapter of Aguiar et al. (2016) surveys key extensions of the model, including the possibility of self-fulfilling crises.

2. Models with renegotiation and recovery generally build on the frameworks of Yue (2010) or Pitchford and Wright (2012). Recent examples include Mihalache (2020) and Dvorkin et al. (2021).

3. Rendahl (2022) discusses the extent to which similar results can be obtained in discrete time by adapting the sparse matrix methods at the core of the solution for continuous time.

inflation as a mechanism for reducing the real value of the debt resembles the notion of partial default studied here, with the caveat that inflation lowers the real value of the entire stock of outstanding debt whereas partial default applied only to the debt service flow.

We propose a computational algorithm by following precedents in the default literature, Bornstein (2020) and Hurtado, Nuño, and Thomas (2023), who adapt the methods introduced by Achdou et al. (2022) to models with full default. Unlike them, we find that the partial default model does not require the formulation and solution of optimal stopping time problems, with associated HJB variational inequalities, as there are no discrete transitions between the regimes of normal market access and market exclusion during default. With partial default, there is a single value function and an unique set of policy functions, possibly discontinuous, which characterize equilibrium outcomes at all times. This greatly simplifies the theoretical and numerical analysis of the model and provides a natural setting for further extensions.

The literature has not restricted attention to quantitative models. Continuous time methods have also been at the forefront of more theoretical analyses of default, including work on debt maturity and multiplicity of equilibria by Aguiar and Amador (2020) and reputation by Amador and Phelan (2021, 2023).

Our model successfully replicates key findings of Arellano, Mateos-Planas, and Ríos-Rull (2023). The impulse response of our model exhibits a hum-shaped pattern in both partial default and debt under a negative income shock. Moreover, our simulation results suggest that, when regaining market access, the sovereign typically carries a higher level of debts than at the point of entering the default episode. Because partial default is an easier way to raise funds, the sovereign tends to rely more on partial defaults than on new issuance of debts, leading to a rapid accumulation of arrears during default episodes. The accumulation of debts causes the sovereign to optimally choose a higher default intensity which results in a protracted default episode. We also quantitatively check the possibility of voluntary restructuring under the partial default setup.

The rest of the paper proceeds as follows. Section 2 presents the model, characterizes policies and bond prices, and defines the equilibrium, Section 3 delivers the quantitative analysis of the model, and Section 4 concludes. Appendices include proofs and derivations omitted from the main text, a discussion of our numerical algorithm, and additional tables and figures.

2 Model

We formulate a continuous time version of the partial default, real endowment model of Arellano, Mateos-Planas, and Ríos-Rull (2023), henceforth Arellano, Mateos-Planas, and Ríos-Rull (2023). The economy consists of a risk-averse sovereign and a unit measure of competitive, risk-neutral international investors. We describe their problems in turn and define the equilibrium.

2.1 The Sovereign

The sovereign of a small, open economy receives an endowment stream, can borrow internationally, and can choose what fraction of the its owed debt service it wants to pay.[4] The part of the debt service payment that the sovereign chooses not to make accumulates as arrears, increasing the stock of debt.

The sovereign's objective is given by discounted lifetime consumption, $\mathbb{E}_0 \int_0^\infty e^{-\rho t} u\left(c_t\right) dt$, where ρ is the sovereign's discount rate. Its stock of debt is denoted by B_t. This long-term debt matures at rate δ and each outstanding unit calls for a coupon payment λ, so that the debt service scheduled by the outstanding debt is given by $\left(\delta+\lambda\right) B_t$.[5] The sovereign chooses a share $d_t \in [0,1]$ of the debt service which it does not pay, and instead accumulates a multiple of it as arrears. Each unit of payment not made increases the stock of debt by κ units. The magnitude of the κ parameter controls the *haircut* suffered by lenders.

The sovereign's flow budget constraint is given by

$$c_t = \phi\left(d_t, z_t\right) e^{z_t} - \left(1-d_t\right)\left(\delta+\lambda\right) B_t + q_t \ell_t. \tag{1}$$

The three terms on the right hand side correspond to the sovereign's income ($\phi\left(d_t, z_t\right) e^{z_t}$), the share of the debt service payment made ($\left(1-d_t\right)\left(\delta+\lambda\right) B_t$), and the proceeds from the sale of new bond units ($q_t \ell_t$), respectively.

The sovereign's income is the product of two components, an exogenous endowment process e^{z_t} and a default penalty function $\phi(d_t, z_t)$. We assume that z_t follows a Ornstein–Uhlenbeck process $dz_t = -\mu z_t dt + \sigma dW_t$ with reflecting barriers at $\underline{z}$ and $\overline{z}$. We include a brief discussion of the compound Poisson process case in the Appendix. The penalty function is decreasing and concave in d_t, and satisfies $\phi(0, z_t) = 1$ and $\phi(1, z_t) > 0$, for any z_t. We assume the presence of a fixed cost, so that default of any intensity leads to a discrete loss of income. As a consequence, $\phi(d_t, z_t)$ is not differentiable at $d_t = 0$. Following Arellano, Mateos-Planas, and Ríos-Rull (2023), we allow for the cost to be a function of z_t as well, and assume $\phi(d_t, z_t)$ is weakly decreasing in z_t, default is more costly when the endowment is higher.

The sovereign issues ℓ_t units of new debt and receives a market price q_t, to be determined by the bond pricing problem of the lenders. Unlike traditional sovereign default models with a discrete choice of default, the sovereign never loses access to financial markets and is able to issue new bond units at any time, albeit at a price which reflects the prospects for repayment going forward. When ℓ_t is negative, we interpret it as the sovereign buying back outstanding instruments in secondary markets.

4. We maintain the standard assumptions of centralized borrowing and centralized default, so that we do not make explicit the behavior of the private sector and the policy instruments which enable the sovereign to implement its preferred outcomes. For the case of decentralized borrowing, see Kim and Zhang (2012).

5. This approach to long-term debt is the continuous time counterpart of the structure employed by standard default models, in order to economy on state variables, for example Hatchondo and Martinez (2009), Chatterjee and Eyigungor (2012), or Hatchondo, Martinez, and Sosa-Padilla (2016).

The evolution of the stock of debt is governed by the drift

$$\frac{dB_t}{dt} = \ell_t - \delta B_t + \kappa\,(\delta+\lambda)\,d_t\,B_t, \tag{2}$$

where the three terms on the right hand side capture the change in debt due to new issuance (ℓ_t), the amortization of maturing debt (δB_t), and the accumulation of arrears, if the sovereign chose to default on some of the owed debt service payments ($\kappa\,(\delta+\lambda)\,d_t\,B_t$).

It will be useful to substitute out the new issuance from the sovereign's flow constraint into the drift of debt, to obtain a new expression for the drift, which we denote by S, as

$$S(B_t, z_t, c_t, d_t, q_t) \equiv \frac{c_t - \phi\,(d_t, z_t)\,e^{z_t}}{q_t} + \left[\left(\frac{1}{q_t} + \left(\kappa - \frac{1}{q_t}\right)d_t\right)(\delta+\lambda) - \delta\right]B_t.$$

Given the initial stock of debt B_0 and the endowment level z_0, the sovereign's problem is summarized by

$$V(B_0, z_0) = \max_{\{c_t, d_t\}_{t\in[0,\infty]}} \mathbb{E}_0\left\{\int_0^\infty e^{-\rho t}u(c_t)dt\right\}$$
$$\text{s.t.} \quad \frac{dB_t}{dt} = S(B_t, z_t, c_t, d_t, q_t)$$

with an associated HJB equation given by

$$\rho V(B,z) = \max_{c,\,d\in[0,1]}\left\{u(c) + S(B,z,c,d,q)V_B(B,z) - \mu z V_z(B,z) + \frac{\sigma^2}{2}V_{zz}(B,z)\right\}. \tag{3}$$

Throughout, B and z subscripts denote the partial derivative with respect to the state variable.

First-order Conditions. We characterize the solution to the maximization problem on the right hand side of the HJB equation with the help of first-order conditions. For consumption, c, the condition is given by $u_c(c) + \frac{V_B(B,z)}{q(B,z)} = 0$, which allows us to express optimal consumption as a function of the bond price schedule and the value function, as

$$c^*(B,z) = u_c^{-1}\left(-\frac{V_B(B,z)}{q(B,z)}\right). \tag{4}$$

If the optimal choice of default intensity is interior, a point to which we return momentarily, it must satisfy $[e^z\phi_d(d,z) - (\kappa\,q(B,z)-1)(\delta+\lambda)B]\;V_B(B,z) = 0$ which can be inverted for the optimal intensity, accounting for the upper bound $d \leq 1$,

$$d^*_{\text{int}}(B,z) = \min\left\{1,\;\phi_d^{-1}\left((\kappa\,q(B,z)-1)\,(\delta+\lambda)\frac{B}{e^z},\,z\right)\right\}. \tag{5}$$

As the $\phi(d,z)$ function is discontinuous at $d = 0$, the optimal default intensity is either $d^*(B,z) = 0$ or $d^*(B,z) = d^*_{\text{int}}(B,z)$, depending on which of these two options maximizes the right hand side of the HJB equation. This reduced to the choice with the smallest associated drift S, given c, as V_B is negative:

$$d^*(B,z) = \begin{cases} d^*_{\text{int}}(B,z) & \text{, if } S(B,z,c,d^*_{\text{int}}(B,z),q)V_B(B,z) \geq S(B,z,c,0,q)V_B(B,z) \\ 0 & \text{, otherwise} \end{cases} \tag{6}$$

The comparative statics of the optimal interior default intensity are summarized by the following Proposition, the proof of which is relegated to the Appendix:

Proposition 1. *The optimal interior default intensity d^*_{int} is a) weakly increasing in the level of debt B, b) weakly decreasing in the bond price q, and c) weakly decreasing in the endowment level z.*

2.2 International Lenders

A unit measure of deep-pocketed, competitive, risk-neutral international lenders buy and hold the sovereign's bonds. They have an outside option given by a risk-free rate r. At any point in time, each unit of debt held by the investors will generate a flow payment $(\delta+\lambda)(1-d)$ and arrears in the form of new debt units, given by $\kappa(\delta+\lambda)d$. The unit price which allows lenders to break even is given by

$$q_t = \mathbb{E}_t \int_t^{\infty} e^{-(r+\delta)(s-t)+\int_t^s \kappa(\delta+\lambda)d_\tau d\tau}(\delta+\lambda)(1-d_s)d_s.$$

Applying Feynman-Kac, the bond price satisfies

$$\xi(d)q(B,z) = (1-d)(\lambda+\delta) + \tilde{S}(B,z)q_B(B,z) - \mu z q_z(B,z) + \frac{\sigma^2}{2}q_{zz}(B,z), \tag{7}$$

where $\tilde{S}(B,z) \equiv S(B,z,c(B,z),d(B,z),q(B,z))$ is the equilibrium drift of the B state and $\xi(d) \equiv r+\delta-\kappa(\delta+\lambda)d$ is the effective discounting for individual bond units, which we introduce to simplify notation. It reflects the lenders' opportunity cost (r), the maturity structure of the debt (δ), and the arrears induced by default. While the evolution of the overall stock of debt is government by the equilibrium drift $\tilde{S}(B,z)$, the cash flow of a single outstanding unit is shaped by $\xi(d)$, which account for default intensity and the accumulation of arrears, but which does not include new issuance by the sovereign.

Spreads and Debt Duration. Before defining the equilibrium, it is useful to introduce several measures. The default risk-free bond price is given by $q^{\text{rf}} = \int_0^\infty e^{-(r+\delta)t}\,(\delta+\lambda)\,dt = \frac{\delta+\lambda}{\delta+r}$ while the yield to maturity implied by the bond price q_t is $r_t = \frac{\delta+\lambda}{q_t} - \delta$.

The spread is the difference between the yield to maturity of the risky bond, subject to partial default, and the corresponding risk-free instrument, $\text{spread}_t = r_t - r = \frac{\delta+\lambda}{q_t} - (\delta+r)$.

Finally, the risk-free Macaulay duration of the bond is $D = \frac{1}{q^{\text{rf}}}\int_0^\infty e^{-(r+\delta)t}(\delta+\lambda)t\,dt = \frac{1}{\delta+r}$.

2.3 Equilibrium

A *Markov Perfect Equilibrium* consist of

a) the sovereign's value function $V(B,z)$,

b) policy functions for consumption and default, $c^*(B,z)$ and $d^*(B,z)$, and

c) the bond price function $q(B,z)$,

such that

1. given $q(B,z)$ and $V(B,z)$, the policies satisfy FOCs (4) and (5) and condition (6),
2. given $q(B,z)$ and the policies, the sovereign's value satisfies the HJB equation (3). and
3. given policy functions, the bond price function satisfies equation (7).

3 Quantitative Analysis

We now turn to the quantitative analysis of our model. Whenever possible we rely on the parameter values in Arellano, Mateos-Planas, and Ríos-Rull (2023) and we compute our model using first the *upwind finite difference scheme* of Achdou et al. (2022), as in Hurtado, Nuño, and Thomas (2023). Appendix D describes out algorithm and its implementation.

3.1 Calibration

We will follow the functional form assumptions of Arellano, Mateos-Planas, and Ríos-Rull (2023) and impose the constant elasticity of intertemporal substitution utility function

$$u\left(c_t\right) = \begin{cases} \frac{c_t^{1-\nu}}{1-\nu} & \text{if } \nu \neq 1 \\ \log c_t & \text{if } \nu = 1 \end{cases}$$

and the default cost function

$$\phi\left(d_t, z_t\right) = \left(1 - \gamma_0 d_t^{\gamma_1}\right)\left[1 - \left(z_t - \tilde{z}\right)\gamma_2 \mathbb{1}_{\{d_t > 0,\, z_t \geq \tilde{z}\}}\right]$$

with $\gamma_0 > 0$, $\gamma_1 > 1$, and $\gamma_2 > 0$. Any strictly positive default intensity leads to a loss of a share $(z_t - \tilde{z})\gamma_2$ of income, but only if z_t exceeds $\tilde{z}$, plus any loss given by $\gamma_0 d_t^{\gamma_1}$, a convex function of default intensity.

We divide parameters into three sets: first, parameters set to the (continuous time equivalents of) values in Arellano, Mateos-Planas, and Ríos-Rull (2023), second, the parameters of the endowment process, and finally, third, parameters we set as part of a moment-matching exercise. Table 1 compiles the resulting values. The parameters we adopt from Arellano, Mateos-Planas, and

Table 1: Calibration

Parameter	Value	Comment
Adapted from Arellano, Mateos-Planas, and Ríos-Rull (2023)		
ν	2.000	Relative risk aversion
ρ	0.047	Sovereign's discount rate
δ	0.120	Bond duration
κ	0.700	Haircut
r	0.039	International risk-free rate
λ	r	Normalization
Endowment Process		
μ	0.225	Time aggregation to AR(1)
σ	0.075	
Default Penalty		
γ_0	0.020	Moment-matching, Table 2
γ_1	2.000	
γ_2	3.500	
$\bar{z}$	0.015	

Notes: ...

Ríos-Rull (2023) are the coefficient of relative risk aversion ν, the discount rate of the sovereign ρ and the risk-free rate r, the parameter controlling bond duration δ, and the haircut parameter κ. We fix $\lambda = r$ as to normalize the risk-free bond price q^{rf} to 1.

We set the parameters of the endowment process, μ and σ, so that when we simulate our process and time-aggregate it to yearly observations, the resulting annual series is well-fitted by the AR(1) discrete time endowment process of Arellano, Mateos-Planas, and Ríos-Rull (2023).

The final set of 4 parameters, which govern the penalty function $\phi(d_t, z_t)$, is used to replicate key moments in the data: the mean and standard deviation of spreads, the average debt to output ratio, and the ratio of the standard deviation of consumption to that of output. Table 2 reports the fit of the model. The data moments are from Table 4 of Arellano, Mateos-Planas, and Ríos-Rull (2023). The moments of model are computed using the ergodic distribution.

3.2 Equilibrium Policies and Moments

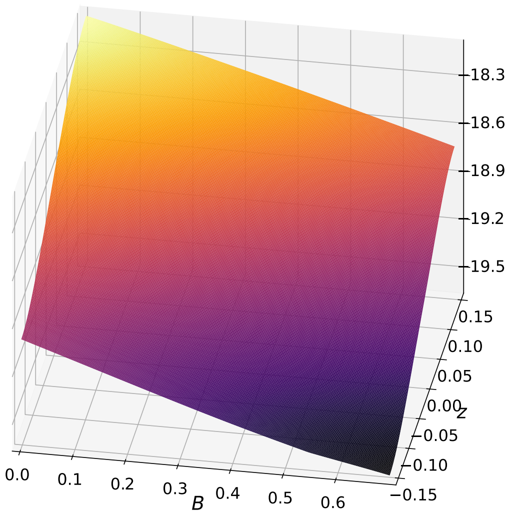
The model's solution under parameterization is given as plotted value function and price function in Figure 1. The value of the sovereign increases in z and decreases in B. We find that the rate of value increase with respect to z is higher than the rate of decrease with respect to B when z is low. Conversely, when z is high, the value changes more sensitively in B than in z. The bond price tends to be low when z is low and B is high, aligning with the sovereign's propensity to opt for higher default intensities under these conditions.

The value function and price function can be better understood with the sovereign's equilibrium policies plotted in Figure 2. The default cost function having a fixed default cost when $z > \bar{z}$

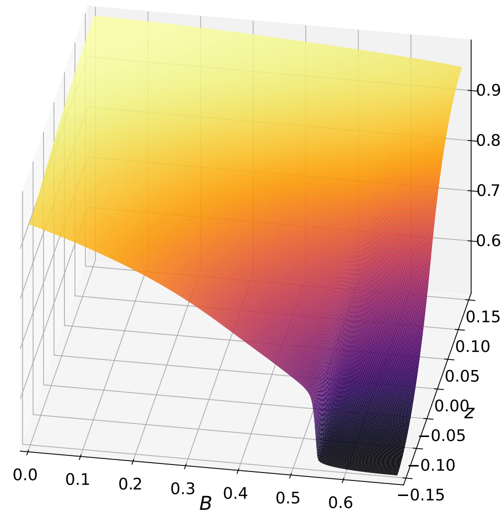
Table 2: Moments in Data and Model

Moment (%)	Data	Model
Partial default		
frequency	0.36	0.56
mean | partial default>0	0.38	0.50
st. dev | partial default>0	0.22	0.16
Debt to output		
mean	0.32	0.40
st. dev	0.18	0.27
Debt service to output		
mean	0.036	0.016
st. dev	0.021	0.024
Debt due to output		
mean	0.049	0.037
Defaulted coupons to output		
mean | partial default>0	0.052	0.049
st. dev | partial default>0	0.064	0.029
Spread		
mean	0.053	0.016
st. dev	0.041	0.001
corr with output	-0.17	-0.91
corr with debt	0.24	0.08
Output		
persistence	0.89	0.89
st. dev	0.10	0.10
Consumption		
st. dev	0.10	0.08

Notes: ...



(a) Value Function $V(B, z)$



(b) Bond Price $q(B, z)$

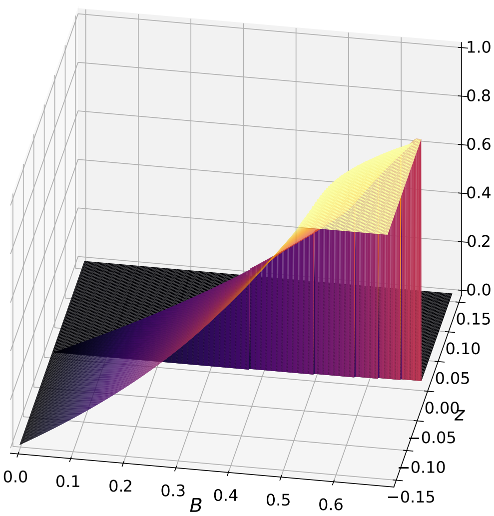
Figure 1: Equilibrium Values and Bond Prices

results in distinct default intensity behavior depending on the value of z. When $z < \bar{z}$, the default cost is $(1 - \gamma_0 d^{\gamma_1})$ and is independent of z. Given the marginal cost of default is 0 when $d = 0$, the sovereign always chooses positive default intensity. The sovereign raises the default intensity until the additional resources released from debt service payment exactly compensate the cost of default. Consequently, the optimal default intensity gradually increases as B increases and reaches complete default when B is high enough. On the other hand, when $z > \bar{z}$, the sovereign pays fixed cost of default, and the fixed cost is proportional to $z - \bar{z}$. The fixed cost of default yields an inaction zone in low-B area, and because the fixed cost is larger for higher z, inaction zone is broader in the high-z region.

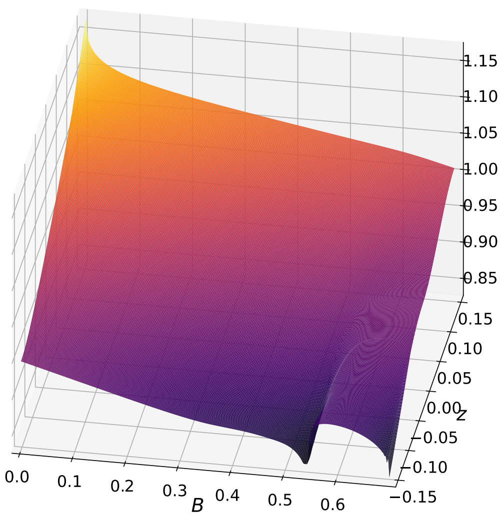
The shape of price function is mainly determined by the optimal default intensity. When z is high and the sovereign does not partially default, the bond price remains relatively stable as increase in B. In contrast, in the low-z region, the default intensity increases sharply, leading to a rapid decline in the bond price.

The overall shapes of optimal default intensity and bond price shape the optimal consumption policy. Due to the fixed cost of default, the sovereign does not use the partial default as borrowing when z is high. Consequently, the consumption drops sharply as the debt service payments are non-defaulted in high B values. For low z, the optimal consumption exhibits a relatively slower decline with respect to B. This is because the sovereign partially defaults and allocates more resources to consumption. However, the bond price shows a steep drop, particularly near the point where the default intensity reaches full default. This sharp drop in bond prices restricts consumption when B is high.

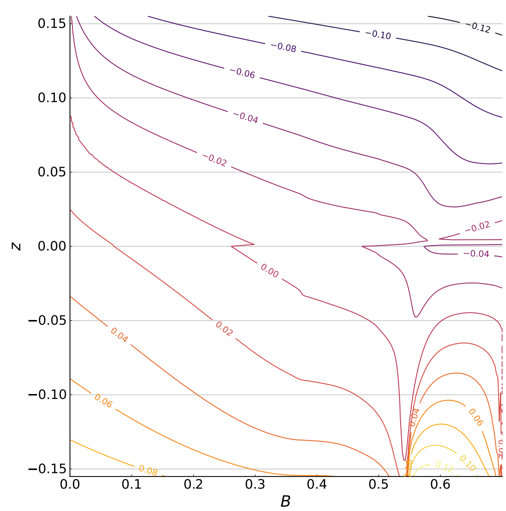
The equilibrium drift of debt obligation, computed based on the bond price, default intensity



(a) Default Intensity $d(B,z)$



(b) Consumption $c(B,z)$



(c) Drift $S(B,z)$

Figure 2: Equilibrium Policy Functions

, and consumption , is illustrated as a contour plot in (c) of Figure 2. The drift represents the rate at which debt obligation is increased or decreased at each point. We find that the drift of debt obligation is positive and high at points where default intensity is high. Specifically, in the region where both B and z are low, total debt obligation accumulates as the sovereign chooses to partially default. The rate of debt accumulation is faster when either B is low or z is low.

Another noteworthy aspect of the figure is the presence of positive drifts in the region of low-z and high-B. In the region, the optimal default intensity is high, leading to a sharp drop in the bond price. After the steep drop, bond prices remain relatively stable at high-B levels, which leads to high consumption and a high drift of debt obligation. We hypothesize that the existence of this positive drift region is why we see lengthy default episodes under the sovereign's partial defaults. Once the region is reached, the sovereign should increase the default intensity, leading to a faster accumulation of arrears, which is followed by another high default intensity.

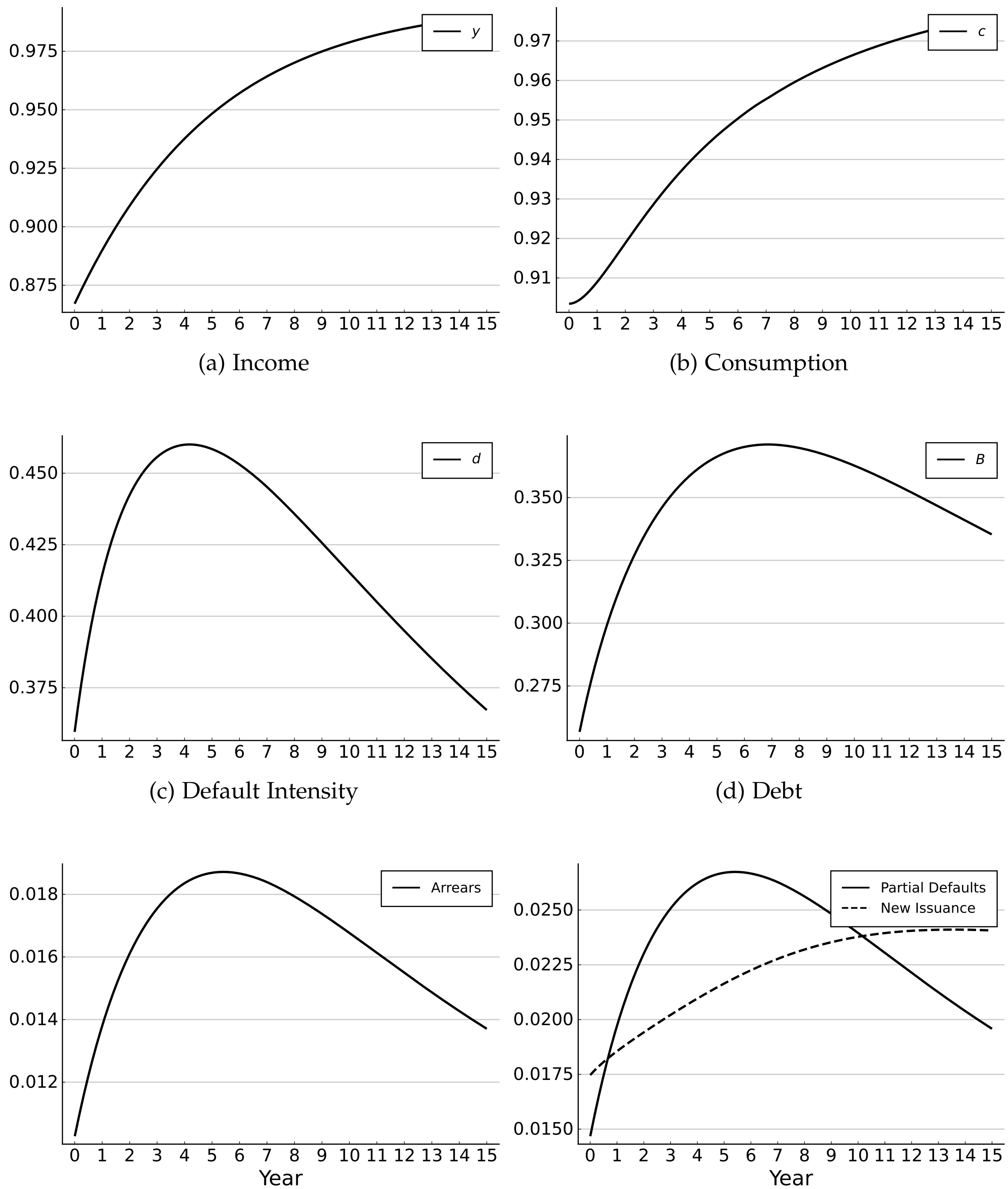
3.3 Impulse Response Functions and Arrears Accumulation

In this subsection, we study impulse response functions of various variables to better understand the model's mechanisms. We simulate on daily basis the impact of a one-period negative shock. The sovereign is initially situated at the mean of ergodic distribution. Figure 3 presents the model's response.

Following the negative shock in the first period, the income begins to rebound. Consumption falls and starts to recover as the income returns back to the steady state, but the fluctuation is smaller due to consumption smoothing. Upon arrival of the negative income shock, The sovereign gradually increases the default intensity as arrears from defaults accumulate to a higher level of debts. The default intensity arrives at the peak in 4 years and decreases afterwards following a significant recovery in the income.

In panel (d) and (e), it can be found that the accumulation of arrears largely shapes the increase in debts. The size of arrears is determined by the default intensity and the size of debts; ($a = \kappa d(\lambda + \delta)B$). In the first 4 years, the sovereign opts for high default intensities, resulting in a high level of arrears. Arrears slightly increase because of the high level of debts even after the sovereign reduces the default intensity. The decline in arrears after 5 years of simulation drives the total debts back to the steady state.

We examine whether the partial default serves as the primary source of borrowing rather than issuing new bonds during the simulation. Panel (f) compares the size of defaulted payments to that of newly issued debts. In the early stage of simulation, new issuance is predominantly used due to the low default intensity, which is a result of a low level of debts. However, as the total debts accumulate, the partial default on the debt due emerges as the primary financing source. New issuance of debts reverts to the primary source of finance only after the income recovers significantly.



(a) Income (b) Consumption

(c) Default Intensity (d) Debt

(e) Arrears (f) Defaulted Payments and Borrowing

Figure 3: Impulse Response Functions

3.4 The Ergodic Distribution and the Length of Default Episodes

One of the advantages of continuous time modeling is easy computation of ergodic distribution. Let $f^*(B,z)$ denote the probability distribution function over states (B,z) in the ergodic distribution. Given the equilibrium policy functions $c^*(B,z)$ and $d^*(B,z)$, we can write the Kolmogorov forward equation(KFE) that governs the stationary equilibrium distribution as follows. The derivation of equation is provided in the Appendix.

$$0 = -\frac{\partial}{\partial B}\left[\tilde{S}(B,z)f^*(B,z)\right] + \frac{\partial}{\partial z}\left[\mu z f^*(B,z)\right] + \frac{\sigma^2}{2}\frac{\partial^2}{\partial z^2}f^*(B,z). \tag{8}$$

The ergodic distribution can be solved directly from (8) without a long run simulation of the model. Moreover, (8) is the adjoint problem of HJB equation (3). In the discretized computational algorithm, the matrix that is used to solve HJB equation is reused to formulate and solve KFE. A more detailed exposition of the computation algorithm can be found in the Appendix as well.

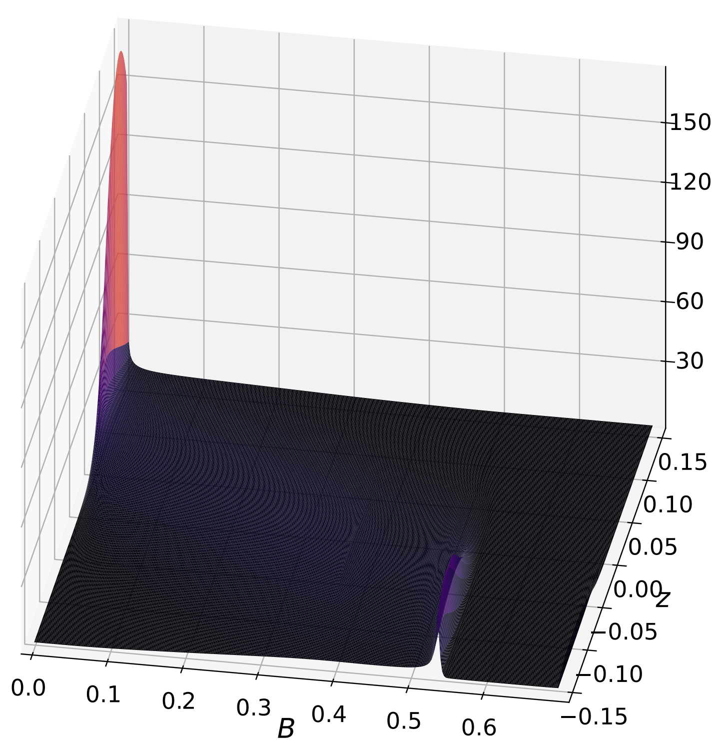
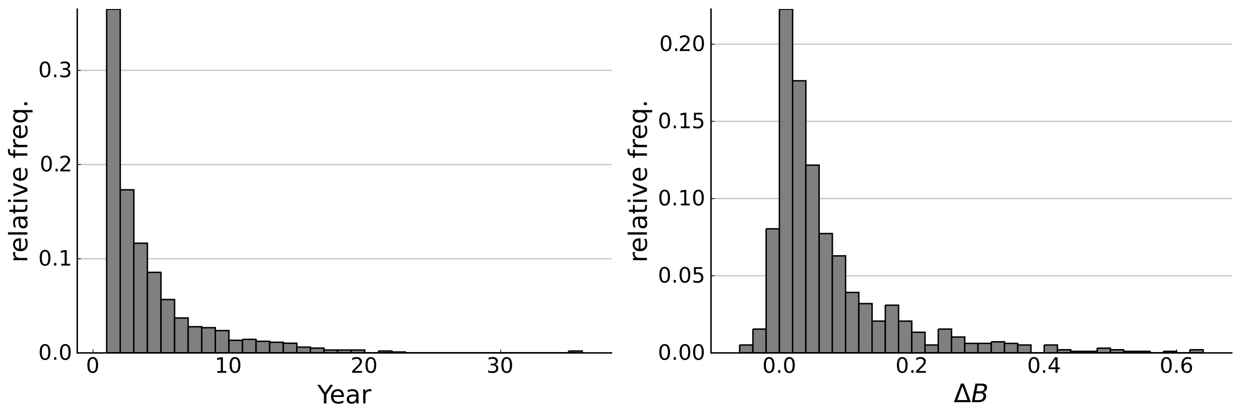


Figure 4: Ergodic Distribution

Figure 4 illustrates the ergodic distribution of the model. We observe a polarized distribution between low-B/high-z and high-B/low-z regions. This distribution is largely shaped by the drift of debt obligation. Because the drift of debt is negative when z is high, a substantial mass is concentrated at low-B/high-z region. For low values of z, the high default intensities of the sovereign causes the debt obligations to drift to a higher level, until a pronounced drop in the bond price occurs. At the points where the quick drop in bond price occurs, the drift turns negative and prevents further growth of debt. Consequently, a large mass is located near the bond price drop region when z is low.



(a) Episode Length

(b) Increase in Debt

Figure 5: Outcomes is Simulated Default Episodes

We conduct a simulation of the model over a span of 10,000 years and analyze the length of default episodes and changes in debt during these episodes. A consecutive series of periods, where default intensities are non-zero at all times and the duration extends beyond a year, is identified as a default scenario. Total 970 episodes are identified in our simulation. Figure 5 (a) shows the distribution of the length of default episodes conditional on the length exceeds a year. Most of episodes terminate in 10 years, but still 8.76% episodes show the length longer than 10 years. The average length of simulated episodes is 4.17 years, and standard deviation is 3.99 years.

We also report the rise in debt levels during the episodes. The distribution is skewed positively, indicating that the debt level increases in most simulated episodes. This is consistent with the observations that the sovereign returns to market access with a higher level of debt than at the time of default. The average increase in debts is 0.074 and the standard deviation is 0.098.

When a substantial size of negative shock arrives, arrears start to accumulate because the sovereign opts for partial default. The sovereign regains access to market when the income recovers enough, but with a higher level of debts.

3.5 Voluntary Restructuring

We examine if there's room for voluntary restructuring of debt. Debtor and creditor can reach a voluntary restructuring if an unexpected reduction of debt level can benefit both parties. Given the sovereign's value decreases in B, such a restructuring can happen if the reduction of debt can improve the total market value of debt $q(B, z) \times B$.

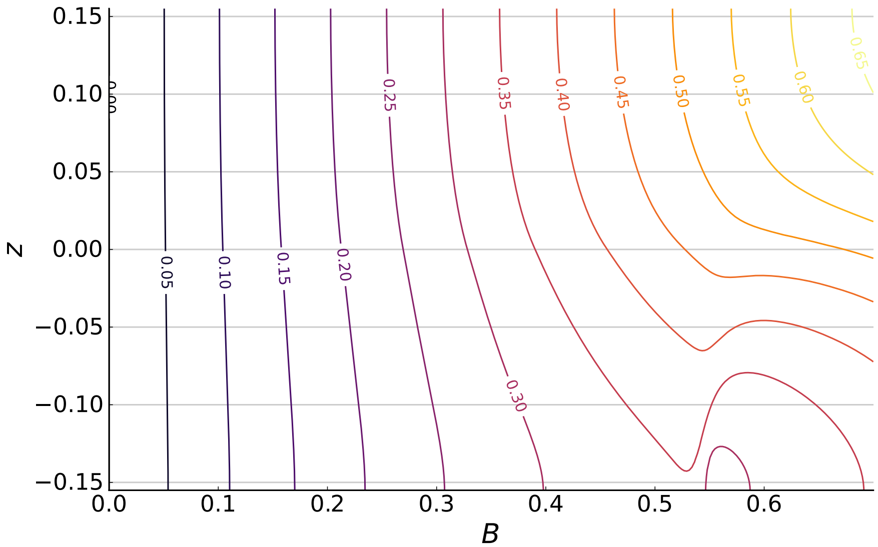


Figure 6: Market Value of Debt

As long as the bond price does not decrease sharply, a positive drift of debt enhances the lenders' value. In contrast, if the bond price drops more rapidly than the growth in debt, lenders can agree with a voluntary reduction in debt. As illustrated in Fig 1 (b), there is a significant price drop when z is low but B is high, which creates an opportunity for voluntary restructuring.

Figure 6 shows the indifference curves for risk-neutral international lenders. An upward sloping region of an indifference curve suggests that reducing the total debt obligation is advantageous for international lenders. This implies the feasibility of a one-time, mutually agreeable debt relief policy for the sovereign using partial defaults.

4 Concluding Remarks

We formulated a tractable model of the sovereign partial default in continuous time setup and suggested a computational algorithm for it. Our model successfully replicates the key findings of Arellano, Mateos-Planas, and Ríos-Rull (2023) such as lengthy default episodes and the dynamics of default episode. Given that HJB equation and price equation are written in simple partial differential equations, the continuous time model is a good avenue to apply deep neural network computational methods. This approach enables the incorporation of a larger state space and more complex models, such as voluntary restructuring initiated by the sovereign. We plan to explore theses in future work.

References

Achdou, Yves, Jiequn Han, Jean-Michel Lasry, Pierre-Louis Lions, and Benjamin Moll. 2022. "Income and wealth distribution in macroeconomics: A continuous-time approach." *The Review of Economic Studies* 89 (1): 45–86.

Aguiar, Mark, and Manuel Amador. 2020. "Self-Fulfilling Debt Dilution: Maturity and Multiplicity in Debt Models." *American Economic Review* 110 (9): 2783–2818.

———. 2021. *The economics of sovereign debt and default.* Vol. 1. Princeton University Press.

Aguiar, Mark, and Gita Gopinath. 2006. "Defaultable debt, interest rates and the current account." *Journal of International Economics* 69 (1): 64–83.

Aguiar, Mark, Satyajit Chatterjee, Harold Cole, and Zachary Stangebye. 2016. "Quantitative models of sovereign debt crises." In *Handbook of Macroeconomics,* 2:1697–1755. Elsevier.

Amador, Manuel, and Christopher Phelan. 2021. "Reputation and Sovereign Default." *Econometrica* 89 (4): 1979–2010.

———. 2023. "Reputation and Partial Default." *American Economic Review: Insights* 5 (2): 158–72.

Arellano, Cristina. 2008. "Default risk and income fluctuations in emerging economies." *American Economic Review* 98 (3): 690–712.

Arellano, Cristina, Yan Bai, and Gabriel Mihalache. 2023. "Deadly Debt Crises: COVID-19 in Emerging Markets." *The Review of Economic Studies* forthcoming (January).

Arellano, Cristina, Xavier Mateos-Planas, and José-Víctor Ríos-Rull. 2023. "Partial default." *Journal of Political Economy* 131 (6): 000–000.

Bornstein, Gideon. 2020. "A continuous-time model of sovereign debt." *Journal of Economic Dynamics and Control* 118:103963.

Chatterjee, Satyajit, and Burcu Eyigungor. 2012. "Maturity, indebtedness, and default risk." *American Economic Review* 102 (6): 2674–2699.

Dvorkin, Maximiliano, Juan M Sánchez, Horacio Sapriza, and Emircan Yurdagul. 2021. "Sovereign debt restructurings." *American Economic Journal: Macroeconomics* 13 (2): 26–77.

Eaton, Jonathan, and Mark Gersovitz. 1981. "Debt with potential repudiation: Theoretical and empirical analysis." *The Review of Economic Studies* 48 (2): 289–309.

Hatchondo, Juan Carlos, and Leonardo Martinez. 2009. "Long-duration bonds and sovereign defaults." *Journal of international Economics* 79 (1): 117–125.

Hatchondo, Juan Carlos, Leonardo Martinez, and Cesar Sosa-Padilla. 2016. "Debt dilution and sovereign default risk." *Journal of Political Economy* 124 (5): 1383–1422.

Hurtado, Samuel, Galo Nuño, and Carlos Thomas. 2023. "Monetary policy and sovereign debt sustainability." *Journal of the European Economic Association* 21 (1): 293–325.

Kim, Yun Jung, and Jing Zhang. 2012. "Decentralized borrowing and centralized default." *Journal of International Economics* 88 (1): 121–133.

Mihalache, Gabriel. 2020. "Sovereign default resolution through maturity extension." *Journal of International Economics* 125:103326.

Pitchford, Rohan, and Mark LJ Wright. 2012. "Holdouts in sovereign debt restructuring: A theory of negotiation in a weak contractual environment." *The Review of Economic Studies* 79 (2): 812–837.

Rendahl, Pontus. 2022. "Continuous vs. discrete time: Some computational insights." *Journal of Economic Dynamics and Control* 144:104522.

Tourre, Fabrice. 2017. *A macro-finance approach to sovereign debt spreads and returns.* The University of Chicago.

Uribe, Martin, and Stephanie Schmitt-Grohé. 2017. *Open economy macroeconomics.* Princeton University Press.

Yue, Vivian Z. 2010. "Sovereign default and debt renegotiation." *Journal of international Economics* 80 (2): 176–187.

Appendix to
"Sovereign Partial Default in Continuous Time"

by Sangdong Kim and Gabriel Mihalache

A Omitted Derivations and Proofs

Derivation of HJB equation (3). The sovereign's problem is

$$V(B_0, z_0) = \max_{\{c_t, d_t\}_{t\in[0,\infty]}} \mathbb{E}_0 \left\{ \int_0^\infty e^{-\rho t} u(c_t) dt \right\}$$
$$\text{s.t.} \quad \frac{dB_t}{dt} = S(B_t, z_t, c_t, d_t, q_t)$$

Note that with small enough time interval Δt,

$$\begin{aligned} V(B_0, z_0) &= \max_{\{c_t, d_t\}_{t\in[0,\infty]}} \mathbb{E}_0 \left\{ \int_0^\infty e^{-\rho t} u(c_t) dt \right\} \\ &= \max_{\{c_t, d_t\}_{t\in[0,\infty]}} \mathbb{E}_0 \left\{ u(c_0)\Delta t + e^{-\rho \Delta t} V(B_{\Delta t}, z_{\Delta t}) \right\} \end{aligned} \tag{9}$$

Taylor expansion of $\mathbb{E}_0 V(B_{\Delta t}, z_{\Delta t})$:

$$\begin{aligned} \mathbb{E}_0 V(B_{\Delta t}, z_{\Delta t}) &= V(B_0, z_0) + V_B(B_0, z_0)\Delta b + V_z(B_0, z_0)\mathbb{E}_0 \Delta z + \frac{1}{2} V_{zz}(B_0, z_0)\mathbb{E}_0(\Delta z)^2 + o(\Delta t) \\ &= V(B_0, z_0) + V_B(B_0, z_0) S(B_0, z_0, c_0, d_0, q_0)\Delta t - \mu z_0 V_z(B_0, z_0)\Delta t + \frac{\sigma^2}{2} V_{zz}(B_0, z_0)\Delta t + o(\Delta t) \end{aligned} \tag{10}$$

Plugging (10) into (9) and taking limitation of $\Delta t \to 0$ gives (3).

Proof of Proposition 1. The proof is straightforward from the decreasing property of $\phi(d, z)$. For any $B_1 \geq B_2$, d_1 and d_2 that satisfies

$$e^z \phi_d(d_i, z) - (\kappa q - 1)(\delta + \lambda) B_i = 0, \quad i = 1, 2$$

should also satisfy $d_1 \geq d_2$ since $\phi_d(d, z) \leq 0$. Then, $d^*_{\text{int}}(B_1, z) = \min\{1, d_1\} \geq \min\{1, d_2\} = d^*_{\text{int}}(B_1, z)$. The proof is similar for *b)* and *c)*.

Derivation of q equation (7). The price of one unit of real bond is

$$q(B_t, z_t) = \mathbb{E}_t \int_t^\infty e^{-(r+\delta)(s-t) + \int_t^s \kappa(\delta+\lambda) d_\tau d\tau} (\delta + \lambda)(1 - d_s) d_s.$$

Again, fix a small enough time interval Δt.

$$q(B_t, z_t) = \int_t^{t+\Delta t} e^{-(r+\delta)(s-t)+\int_t^s \kappa(\delta+\lambda)d_\tau d\tau}(\delta+\lambda)(1-d_s)d_s + e^{-(r+\delta)\Delta t+\int_t^{t+\Delta t}\kappa(\delta+\lambda)d_\tau d\tau} q(B_{t+\Delta t}, z_{t+\Delta t})$$
$$\approx (\delta+\lambda)(1-d_t)\Delta t + e^{-[r+\delta-\kappa d(\delta+\lambda)]\Delta t} q(B_{t+\Delta t}, z_{t+\Delta t})$$

Define $\xi(d) = r + \delta - \kappa d(\delta+\lambda)$ for notational simplicity. By the same Taylor expansion as in (10) and taking limitation of $\Delta t \to 0$, we get (7).

Derivation of KFE (8). Let $\mathbf{x} = (B, z)$ and $\Delta\mathbf{x} = (\Delta B, \Delta z)$, where $\Delta B = \tilde{S}(\mathbf{x})\Delta T, \Delta z = -\mu z \Delta t + \sigma d W_{\Delta t}$ for a short time interval Δt.

$f(\mathbf{x}, t)$ is the pdf in time t. The pdf evolves over time by the following equation.

$$f(\mathbf{x}, t+\Delta t) = f(\mathbf{x}, t) + \int_{\Delta\mathbf{x}} T_t(\mathbf{x}-\Delta\mathbf{x}, \Delta\mathbf{x}) f(\mathbf{x}-\Delta\mathbf{x}, t) d\mathbf{x}_1 d\mathbf{x}_2 - \int_{\Delta\mathbf{x}} T_t(\mathbf{x}, \Delta\mathbf{x}) f(\mathbf{x}, t) d\mathbf{x}_1 d\mathbf{x}_2$$

Here, $T_t(\mathbf{x}, \mathbf{y})$ is the rate of transition from $\mathbf{x}$ to $\mathbf{x}+\mathbf{y}$ at a point in time t. The pdf at point $\mathbf{x}$ in time $t+\Delta t$ should be the pdf at the same point in time t plus the sum of all inflows less the sum of all outflows.

Fix a vector $\Delta\mathbf{x}$. With the fixed $\Delta\mathbf{x}$, define a function $g_{\Delta\mathbf{x}}(\mathbf{x}, t)$ by

$$g_{\Delta\mathbf{x}}(\mathbf{x}, t) = T_t(\mathbf{x}, \Delta\mathbf{x}) f(\mathbf{x}, t).$$

The Taylor expansion of $g_{\Delta\mathbf{x}}(\mathbf{x}, t)$ gives

$$\begin{aligned}
g_{\Delta\mathbf{x}}(\mathbf{x}-\Delta\mathbf{x}, t) = {} & T_t(\mathbf{x}-\Delta\mathbf{x}, \Delta\mathbf{x}) f(\mathbf{x}-\Delta\mathbf{x}, t) \\
= {} & T_t(\mathbf{x}, \Delta\mathbf{x}) f(\mathbf{x}, t) + (-1)\times\Delta B \frac{\partial}{\partial B}\left[T_t(\mathbf{x}, \Delta\mathbf{x}) f(\mathbf{x}, t)\right] + (-1)\times \Delta z \frac{\partial}{\partial z}\left[T_t(\mathbf{x}, \Delta\mathbf{x}) f(\mathbf{x}, t)\right] \\
& + \frac{1}{2}(\Delta B)^2 \frac{\partial^2}{\partial B^2}\left[T_t(\mathbf{x}, \Delta\mathbf{x}) f(\mathbf{x}, t)\right] + \frac{1}{2}(\Delta z)^2 \frac{\partial^2}{\partial z^2}\left[T_t(\mathbf{x}, \Delta\mathbf{x}) f(\mathbf{x}, t)\right] \\
& + (\Delta B)(\Delta z)\frac{\partial^2}{\partial B \partial z}\left[T_t(\mathbf{x}, \Delta\mathbf{x}) f(\mathbf{x}, t)\right] + h.o.t
\end{aligned}$$

Plugging this into the first equation gives

$$\begin{aligned}
\frac{f(\mathbf{x}, t+\Delta t) - f(\mathbf{x}, t)}{\Delta t} = {} & \frac{1}{\Delta t}\int_{\Delta\mathbf{x}} (-1)\times\Delta B \frac{\partial}{\partial B}\left[T_t(\mathbf{x}, \Delta\mathbf{x}) f(\mathbf{x}, t)\right] d\mathbf{x}_1 d\mathbf{x}_2 \\
& + \frac{1}{\Delta t}\int_{\Delta\mathbf{x}} (-1)\times\Delta z \frac{\partial}{\partial z}\left[T_t(\mathbf{x}, \Delta\mathbf{x}) f(\mathbf{x}, t)\right] d\mathbf{x}_1 d\mathbf{x}_2 \\
& + \frac{1}{\Delta t}\int_{\Delta\mathbf{x}} \frac{1}{2}(\Delta B)^2 \frac{\partial^2}{\partial B^2}\left[T_t(\mathbf{x}, \Delta\mathbf{x}) f(\mathbf{x}, t)\right] d\mathbf{x}_1 d\mathbf{x}_2 \\
& + \frac{1}{\Delta t}\int_{\Delta\mathbf{x}} \frac{1}{2}(\Delta z)^2 \frac{\partial^2}{\partial z^2}\left[T_t(\mathbf{x}, \Delta\mathbf{x}) f(\mathbf{x}, t)\right] d\mathbf{x}_1 d\mathbf{x}_2 \\
& + \frac{1}{\Delta t}\int_{\Delta\mathbf{x}} (\Delta B)(\Delta z) \frac{\partial^2}{\partial B \partial z}\left[T_t(\mathbf{x}, \Delta\mathbf{x}) f(\mathbf{x}, t)\right] d\mathbf{x}_1 d\mathbf{x}_2 + h.o.t
\end{aligned}$$

Integration $\int_{\Delta \mathbf{x}}$ can flip the order with partial differentiation. We use

$$\Delta B = \tilde{S}(B, z)\Delta t,$$
$$\Delta z = -\mu z \Delta t + \sigma dW_{\Delta t},$$
$$(\Delta z)^2 = \sigma^2 \Delta t.$$

When Δt is sufficiently small, ignoring the terms of Δt order higher than 1,

$$\frac{\partial}{\partial t} f(\mathbf{x}, t) = -\frac{\partial}{\partial B}\left[\tilde{S}(\mathbf{x}) f(\mathbf{x}, t)\right] + \frac{\partial}{\partial z}\left[\mu z f(\mathbf{x}, t)\right] + \frac{\sigma^2}{2}\frac{\partial^2}{\partial z^2} f(\mathbf{x}, t)$$

In the stationary distribution $f_t(\mathbf{x}, t)$ should be zero. This gives the KFE for stationary distribution

$$0 = -\frac{\partial}{\partial B}\left[\tilde{S}(\mathbf{x}) f(\mathbf{x})\right] + \frac{\partial}{\partial z}\left[\mu z f(\mathbf{x})\right] + \frac{\sigma^2}{2}\frac{\partial^2}{\partial z^2} f(\mathbf{x}).$$

B The Compound Poisson Process Case

In the main text, we restricted attention to an Ornstein–Uhlenbeck endowment process z_t. We have also solved our model for the case of jumps, where z_t switched with a constant intensity between two or more discrete levels, with similar qualitative properties. We conjecture that our environment can accommodate an endowment process with both jumps and a diffusion component.

In this Appendix, we lay out the result of the model with jump process of endowment. We model z as a switch between two different levels of endowment. The switch occurs at Poisson arrival rate of η. Every other model component remains intact as in the main text. The HJB equation of the sovereign is

$$\rho V(B, z_i) = \max_{c,d}\; u(c) + S(B, z, c, d, q)V_B + \eta\left(V(B, z_{-i}) - V(B, z_i)\right); \quad i = 1, 2.$$

We solve for the sovereign's value function using the implicit upwind finite difference scheme the detailed exposition of which is presented in Appendix D.

Figure 7 shows the computational results. The sovereign's value decreases in B and increases in z. Similarly, the bond price decreases in B and increases in z. For the optimal default intensity, when z is lower than the threshold $\bar{z}$, which represents no fixed cost of default penalty, the default intensity gradually increases as B increases. However, for a higher level of z, the sovereign refrains from defaulting until a certain level of debts is reached due to the fixed cost of default. Beyond this point, the default intensity jumps up to a significant level and continues to gradually increase when B increases. The bond price shows a sharp decline near the point the sovereign completely defaults. The bond price schedule and default intensity together determine the optimal consumption. When the drop of bond price is steep, the sovereign reduces the consumption and debt obligations, leading to a lower drift. Conversely, when the marginal drop in bond price is small, the sovereign recovers the consumption, resulting in a higher debt drift.

C Discrete Time Case

$$
\begin{aligned}
V(B,z) = \max_{B',c,d} u(c) + \beta \mathbb{E}_{z'|z} V(B',z') \\
\text{s.t. } c = \phi(d,z)e^z - (1-d)(\delta+\lambda)B + q(B',z)\ell \\
B' = (1-\delta)B + \kappa(\delta+\lambda)dB + \ell
\end{aligned} \tag{11}
$$

$$
\phi_d(d,z)e^z + (\delta+\lambda)B - q(B',z)\kappa(\delta+\lambda)B = 0 \tag{Interior FOC d}
$$

$$
\phi_d(d^{\text{int}},z) = [\kappa q(B',z) - 1](\delta+\lambda)\frac{B}{e^z}
$$

$$
\begin{cases} c^{\text{int}} = \phi(d^{\text{int}},z)e^z - (1-d^{\text{int}})(\delta+\lambda)B + q(B',z)(B' - (1-\delta+\kappa(\delta+\lambda)d^{\text{int}})B), & \text{if } d = d^{\text{int}} \\ c^0 = e^z - (\delta+\lambda)B + q(B',z)(B' - (1-\delta)B), & \text{otherwise} \end{cases} \tag{12}
$$

$$
q(B',z) = \frac{1}{1+r}\mathbb{E}_{z'|z}\left\{(\delta+\lambda)(1-\mathbb{D}(B',z')) + [1-\delta+\kappa(\delta+\lambda)\mathbb{D}(B',z')]q(\mathbb{B}(B',z'),z')\right\} \tag{13}
$$

D Algorithm

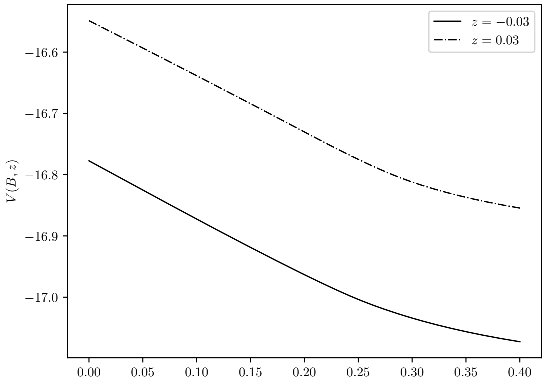
D.1 The Implicit Upwind Finite Difference Scheme

Solution to HJB equation (3). We compute our model using the implicit upwind finite difference scheme of Achdou et al. (2022), following Hurtado, Nuño, and Thomas (2023)'s application to their discrete default model. Here, we provide details about the computation algorithm of solving (3) and (7).

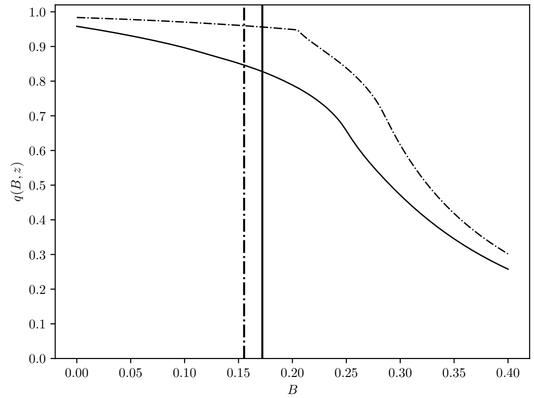
We descritize B state space into n grid points, z state space into m grid points and use the notation $V_{i,j} = V(B_i, z_j)$, $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, m$. The derivative of V with respect to B is numerically computed by either forward difference or backward difference, but the derivative with respect to z is computed using forward difference.

$$
\begin{aligned}
\partial_{B,F} V_{ij} &= \frac{V_{i+1,j} - V_{i,j}}{\Delta B}, \quad \partial_{B,B} V_{i,j} = \frac{V_{i,j} - V_{i-1,j}}{\Delta B} \\
\partial_z V_{i,j} &= \frac{V_{i,j+1} - V_{i,j}}{\Delta z}, \quad \partial_{zz} V_{i,j} = \frac{V_{i,j+1} + V_{i,j-1} - 2V_{i,j}}{(\Delta z)^2}
\end{aligned}
$$

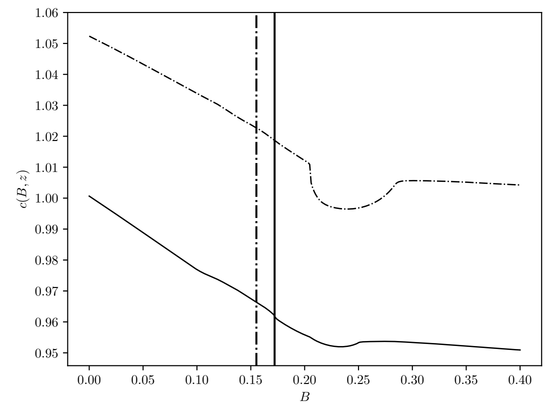
Given $q(B,z)$ and a guess of value function $V^k(B,z)$, we can compute the optimal consumption using forward/backward difference from the first order condition (4). Denote consumption



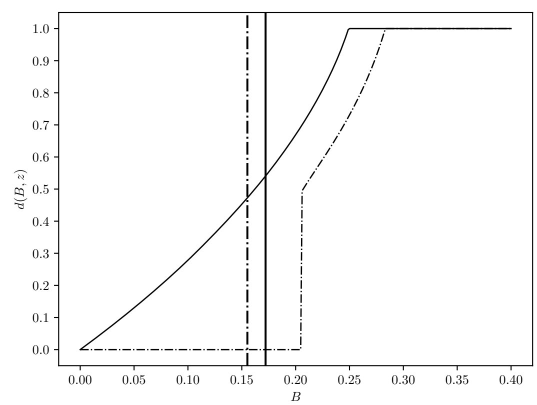
(a) value function



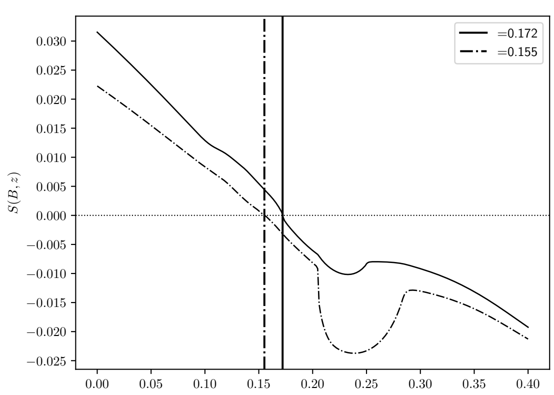
(b) bond price



(c) consumption



(d) default intensity



(e) drift

Figure 7: The Poisson Process Case

Notes: Vertical lines indicate the point where the drift of debts is zero.

computed using forward difference by $c^k_{F,i,j}$ and consumption computed using backward difference by $c^k_{B,i,j}$.

$$c^k_{F,i,j} = u_c^{-1}\left(-\frac{\partial_{B,F}V^k_{i,j}}{q(B,z)}\right), \quad c^k_{B,i,j} = u_c^{-1}\left(-\frac{\partial_{B,B}V^k_{i,j}}{q(B,z)}\right).$$

Note that computation of d^*_{int} does not require numerical approximation of V_B as shown in (5).

$$d^*_{int,i,j} = \min\left\{1, \phi_d^{-1}\left((\kappa q(B_i,z_j)-1)(\delta+\lambda)\frac{B}{e^{z_j}}, z_j\right)\right\}$$

Under each forward difference and backward difference, we determine the optimal default intensity by comparing the flow value (right-hand-side of (3)) of the sovereign when it chooses $d=0$ and $d=d^*_{int}$. Specifically, the choice of default intensity changes the flow value only through a change in the drift of B.

$$d^k_{F,i,j} = \underset{d\in\{0,d^*_{int,i,j}\}}{\arg\max}\ S(B_i,z_j,c^k_{F,i,j},d,q)\partial_{B,F}V^k_{i,j}$$
$$d^k_{B,i,j} = \underset{d\in\{0,d^*_{int,i,j}\}}{\arg\max}\ S(B_i,z_j,c^k_{B,i,j},d,q)\partial_{B,B}V^k_{i,j}$$

Since V_B is negative, we are choosing the optimal default intensity from $\{0, d^*_{int,i,j}\}$ that minimizes the drift of B.

With computed forward/backward optimal consumption and default intensity, we compute the drift of B implied by forward/backward difference of V^n_B.

$$S^k_{F,i,j} = S(B_i,z_j,c^k_{F,i,j},d^k_{F,i,j},q)$$
$$S^k_{B,i,j} = S(B_i,z_j,c^k_{B,i,j},d^k_{B,i,j},q)$$

The upwind scheme is to use the forward difference to compute the derivatives of V wherever $S^k_{F,i,j}$ is positive and to use the backward difference wherever $S^k_{B,i,j}$ is negative. If neither $S^k_{F,i,j} > 0$ nor $S^k_{B,i,j} < 0$, we find the optimal default intensity among $\{0, d^*_{int,i,j}\}$ that maximizes the consumption while imposing zero drift of B. Denote the maximized consumption under zero drift by $c^k_{0,i,j}$.

The upwind finite implicit updating rule is shown in the equation below.

$$V^{k+1}_{i,j} = (1-\rho\Delta t)V^k_{i,j} + \Delta t\Big[\mathbb{1}_{S^k_{F,i,j}>0}\left\{u(c^k_{F,i,j}) + S^k_{F,i,j}\partial_{B,F}V^{k+1}_{i,j}\right\} + \mathbb{1}_{S^k_{B,i,j}<0}\left\{u(c^k_{B,i,j}) + S^k_{B,i,j}\partial_{B,B}V^{k+1}_{i,j}\right\}$$
$$+ \mathbb{1}_{S^k_{i,j}=0}u(c^k_{0,i,j}) - \mu z_j\partial_z V^{k+1}_{i,j} + \frac{\sigma^2}{2}\partial_{zz}V^{k+1}_{i,j}\Big].$$

From forward and backward difference formula,

$$\frac{V_{i,j}^{k+1}-V_{i,j}^{k}}{\Delta t}+\rho V_{i,j}^{k+1}=\Big[\mathbb{1}_{S_{F,i,j}^{k}>0}u(c_{F,i,j}^{k})+\mathbb{1}_{S_{B,i,j}^{k}<0}u(c_{B,i,j}^{k})+\mathbb{1}_{S_{i,j}^{k}=0}u(c_{0,i,j}^{k})\Big]$$
$$+\left[-\frac{\mathbb{1}_{S_{B,i,j}^{k}<0}S_{B,i,j}^{k}}{\Delta B},\quad \frac{\mathbb{1}_{S_{B,i,j}^{k}<0}S_{B,i,j}^{k}-\mathbb{1}_{S_{F,i,j}^{k}>0}S_{F,i,j}^{k}}{\Delta B},\quad \frac{\mathbb{1}_{S_{F,i,j}^{k}>0}S_{F,i,j}^{k}}{\Delta B}\right]\begin{bmatrix}V_{i-1,j}^{k+1}\\ V_{i,j}^{k+1}\\ V_{i+1,j}^{k+1}\end{bmatrix}$$
$$+\left[\frac{\sigma^2}{2(\Delta z)^2},\quad \frac{\mu z_j}{\Delta z}-\frac{\sigma^2}{(\Delta z)^2},\quad -\frac{\mu z_j}{\Delta z}+\frac{\sigma^2}{2(\Delta z)^2}\right]\begin{bmatrix}V_{i,j-1}^{k+1}\\ V_{i,j}^{k+1}\\ V_{i,j+1}^{k+1}\end{bmatrix}$$

In matrix notation,

$$\frac{\mathbf{V}^{k+1}-\mathbf{V}^{k}}{\Delta t}+\rho\mathbf{V}^{k+1}=\mathbf{u}^{k}+\mathbf{A}^{k}\mathbf{V}^{k+1}$$

where

$$\mathbf{V}=\begin{bmatrix}V_{1,1}\\ V_{2,1}\\ \vdots\\ V_{n,1}\\ V_{1,2}\\ \vdots\\ V_{n,m}\end{bmatrix},\quad \mathbf{u}^{k}=\begin{bmatrix}\mathbb{1}_{S_{F,1,1}^{k}>0}u(c_{F,1,1}^{k})+\mathbb{1}_{S_{B,1,1}^{k}<0}u(c_{B,1,1}^{k})+\mathbb{1}_{S_{1,1}^{k}=0}u(c_{0,1,1}^{k})\\ \mathbb{1}_{S_{F,2,1}^{k}>0}u(c_{F,2,1}^{k})+\mathbb{1}_{S_{B,2,1}^{k}<0}u(c_{B,2,1}^{k})+\mathbb{1}_{S_{2,1}^{k}=0}u(c_{0,2,1}^{k})\\ \vdots\\ \mathbb{1}_{S_{F,n,m}^{k}>0}u(c_{F,n,m}^{k})+\mathbb{1}_{S_{B,n,m}^{k}<0}u(c_{B,n,m}^{k})+\mathbb{1}_{S_{n,m}^{k}=0}u(c_{0,n,m}^{k})\end{bmatrix}$$

and $\mathbf{A}^k\in\mathcal{M}(n\times m,n\times m)$ is a sparse matrix. Elements of $\mathbf{A}^k$ for $i=1,2,\cdots n$ and $j=1,2,\cdots m$ are

$$\mathbf{A}^{k}\Big((i-1)m+j,(i-1)m+j-n\Big)=\frac{\sigma^2}{2(\Delta z)^2}$$
$$\mathbf{A}^{k}\Big((i-1)m+j,(i-1)m+j-1\Big)=-\frac{\mathbb{1}_{S_{B,i,j}^{n}<0}S_{B,i,j}^{n}}{\Delta B}$$
$$\mathbf{A}^{k}\Big((i-1)m+j,(i-1)m+j\Big)=\frac{\mathbb{1}_{S_{B,i,j}^{n}<0}S_{B,i,j}^{n}-\mathbb{1}_{S_{F,i,j}^{n}>0}S_{F,i,j}^{n}}{\Delta B}+\frac{\mu z_j}{\Delta z}-\frac{\sigma^2}{(\Delta z)^2}$$
$$\mathbf{A}^{k}\Big((i-1)m+j,(i-1)m+j+1\Big)=\frac{\mathbb{1}_{S_{F,i,j}^{n}>0}S_{F,i,j}^{n}}{\Delta B}$$
$$\mathbf{A}^{k}\Big((i-1)m+j,(i-1)m+j+n\Big)=-\frac{\mu z_j}{\Delta z}+\frac{\sigma^2}{2(\Delta z)^2}$$

The rule of updating value function is a sparse matrix equation.

$$\mathbf{V}^{k+1}=\Big[(1+\rho\Delta t)\,\mathbf{I}-\Delta t\mathbf{A}^{k}\Big]^{-1}\Big[\Delta t\mathbf{u}^{k}+\mathbf{V}^{k}\Big]$$

Solution to q equation (7)**.** We use the same implicit finite difference upwind updating to solve the price equation (7). The sparse matrix constructed in solving HJB equation can be reused here. With the same notations as in HJB equation, the discretized version of updating rule is

$$\begin{aligned}\frac{q_{i,j}^{k+1}-q_{i,j}^{k}}{\Delta t}+\xi(d_{i,j}^{k})q_{i,j}^{k+1}&=(\lambda+\delta)(1-d_{i,j}^{k})\\&+\left[-\frac{\mathbb{1}_{S_{B,i,j}^k<0}S_{B,i,j}^k}{\Delta B},\quad\frac{\mathbb{1}_{S_{B,i,j}^k<0}S_{B,i,j}^k-\mathbb{1}_{S_{F,i,j}^k>0}S_{F,i,j}^k}{\Delta B},\quad\frac{\mathbb{1}_{S_{F,i,j}^k>0}S_{F,i,j}^k}{\Delta B}\right]\begin{bmatrix}q_{i-1,j}^{k+1}\\q_{i,j}^{k+1}\\q_{i+1,j}^{k+1}\end{bmatrix}\\&+\left[\frac{\sigma^2}{2(\Delta z)^2},\quad\frac{\mu z_j}{\Delta z}-\frac{\sigma^2}{(\Delta z)^2},\quad-\frac{\mu z_j}{\Delta z}+\frac{\sigma^2}{2(\Delta z)^2}\right]\begin{bmatrix}q_{i,j-1}^{k+1}\\q_{i,j}^{k+1}\\q_{i,j+1}^{k+1}\end{bmatrix}\end{aligned}$$

where

$$d_{i,j}^k=\mathbb{1}_{S_{F,i,j}^k>0}d_{F,i,j}+\mathbb{1}_{S_{B,i,j}^k<0}d_{B,i,j}+\mathbb{1}_{S_{i,j}^k=0}d_{0,i,j}.$$

This can be rewritten in matrix notation.

$$\frac{\mathbf{q}^{k+1}-\mathbf{q}^k}{\Delta t}+\boldsymbol{\xi}\mathbf{q}^{k+1}=(\lambda+\delta)\mathbf{d}^k+\mathbf{A}^k\mathbf{q}^{k+1},$$

with

$$\begin{aligned}\boldsymbol{\xi}&=\operatorname{diag}(\xi(d_{1,1}),\xi(d_{2,1}),\cdots\xi(d_{n,m}))\in\mathcal{M}(n\times m,n\times m),\\\mathbf{q}&=[q_{1,1},q_{2,1},\cdots,q_{n,m}]^T,\\\mathbf{d}^k&=[1-d_{1,1}^k,1-d_{2,1}^k,\cdots,1-d^k(n,m)]^T,\end{aligned}$$

and the same $\mathbf{A}^k$ as in the computation of HJB equation. The updating rule is

$$\mathbf{q}^{k+1}=\left[(\mathbf{I}+\boldsymbol{\xi}\Delta t)-\Delta t\mathbf{A}^k\right]^{-1}\left[\Delta t(\lambda+\delta)\mathbf{d}^k+\mathbf{q}^k\right]$$

Ergodic distribution Kolmogorov forward equation is the adjoint problem of HJB equation. In other words, if we write HJB equation as follows

$$\rho\mathbf{V}=\mathbf{u}+\mathbf{AV},$$

Kolmogorov forward equation can be written as

$$0=\mathbf{A}^T\mathbf{f}.$$

We compute the ergodic distribution of (B,z) using the above fact. Specifically, if $\mathbf{A}^k$ is the sparse matrix constructed in the last iteration of HJB updating, the ergodic distribution ($\mathbf{f}$) can be found by solving $0 = (\mathbf{A}^k)^T \mathbf{f}$.

Computation Algorithm. The entirety of algorithm is given below.

1. Begin with initial guess $V_{i,j}^k$ and $q_{i,j}^k$.
2. Compute $\{c_{X,i,j}^k, d_{X,i,j}^k, S_{X,i,j}^k\}_{X \in \{F,B,0\}}$ and update the value function to $V_{i,j}^{k+1}$.
3. Update the price function to $q_{i,j}^{k+1}$.
4. If $V_{i,j}^{k+1}$ is close enough to $V_{i,j}^k$ and $q_{i,j}^{k+1}$ is close enough to $q_{i,j}^k$, stop. Otherwise, iterate above with new initial guess $V_{i,j}^{k+1}$ and $q_{i,j}^{k+1}$.
5. Solve Kolmogorov forward equation using $\mathbf{A}$ of the last iteration in value function updating.